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Topology and Fractional Quantum Hall Effect

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Abstract

Starting from Laughlin type wave functions with generalized periodic boundary conditions describing the degenerate groundstate of a quantum Hall system we explictly construct r dimensional vector bundles. It turns out that the filling factor ν is given by the topological quantity $\frac{c_1}{r}$ where c_1 is the first chern number of these vector bundles. In addition, we managed to proof that under physical natural assumptions the stable vector bundles correspond to the experimentally dominating series of measured fractional filling factors $\nu = \frac{n}{2pn \pm 1}$. Most remarkably, due to the very special form of the Laughlin wave functions the fluctuations of the curvature of these vector bundles converge to zero in the limit of infinitely many particles which shows a new mathematical property. Physically, this means that in this limit the Hall conductivity is independent of the boundary conditions which is very important for the observabilty of the effect. Finally we discuss the relation of this result to a theorem of Donaldson.

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1 Introduction

Since its discovery the integer and fractional Quantum Hall effect (IQHE, FQHE) [1, 2] have fascinated both experimentalists and theorists. The conductance of a two dimensional electron gas in a high magnetic field at low temperatures exhibits quantized plateau values of the form $\sigma_{xy} = \frac{e^2}{h}\nu$ where the filling fraction ν is an integer or fractional number. It was very astonishing that the conductance does not depend on the size and geometry of the experimental sample and that the quantization could be measured with such a high precision. In the last few years, it became more and more obvious that in completely spin-polarized quantum Hall systems the fractional values

$$\nu = \frac{n}{2pn \pm 1} \qquad \text{and} \qquad \nu = 1 - \frac{n}{2pn \pm 1} \tag{1.1}$$

first proposed by J.K. Jain [3] turn out to dominate the experimental measurements of the FQHE for $\nu < 1$ [4, 5, 6]. *.

From a theoretical point of view there are two main questions that have to be solved in order to understand the QHE: a) What is the reason for the exact quantization of the Hall conductance especially, for the series (1.1)? b) Why do Hall plateaus occur? The aim of this letter is to give some insight to the first problem, especially to understand the appearance of the experimentally observed main series (1.1) by means of topological arguments. The second question is not considered here, but it is widely suspected that disorder effects are responsible for the occurrence and the width of the Hall plateaus, which are essential for observing the IQHE as well as the FQHE [8].

Historically, first arguments for the exact quantization of the Hall conductance were given by Laughlin who showed that the Hall conductance is quantized whenever the Fermi energy lies in an energy gap, even if the gap lies within a Landau level [9]. A very important contribution to an understanding of the IQHE was provided by Thouless et. al., deriving the Hall conductance from the Kubo formula via linear response theory and proving the locality of the Hall conductance inside the sample; thus, the main consequence of this work is that the Hall conductance is essentially insensitive to the boundary conditions one imposes at the edges of the physical samples [10, 12]. J.E. Avron et. al. and M. Kohmoto performed the topological analysis of the IQHE in a convincing way and showed that the Hall conductance can be represented by the first Chern number of a line bundle over the magnetic Brillouin zone which is an integer-valued topological constant [13, 14, 15, 16]. All these considerations assume that the groundstate is nondegenerate.

In the case of the FQHE there are many completely different theoretical descriptions. The aim of that paper is to show the connection of two of them.

On the one hand, Thouless et. al. required a degeneracy in order to explain the fractional quantization, since a nondegenerate groundstate always leads to an integral quantization. Consequently the Hall conductance has to be expressed by the first Chern number of a vector bundle divided by its rank, where the rank is equal to the degeneracy [11].

*In addition to these fractional values other fractional plateaus are measured but without vanishing of the longitudinal conductivity. This has been explained by J. Fröhlich et. al. [7] by a classification of quantum Hall fluids

On the other hand, an important step was taken by Laughlin who wrote down the wave functions for the fundamental fractions $\nu = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \ldots$ [17]. Extensive calculations have proven these wave functions to be extremely close to the numerical exact solutions [8]. They play a special role in a hierarchal scheme in which a daughterstate is obtained at each step from a condensation of quasiparticles of the parent state into a correlated low energy state [18, 19].

Laughlin's wave functions defined on a disk or sphere do not show any degeneracy of the groundstate. However, this degeneracy naturally occurs if the wave functions are defined on a torus, that means if generalized periodic boundary conditions are imposed. First F.D.M. Haldane and E.H. Rezayi [20] and later E. Keski-Vakkurri and X.G. Wen [21] have constructed Laughlin type wave functions with periodic boundary conditions which we slightly generalized.

This paper is organized as follows: In the next two sections we briefly review the two theoretical descriptions of the FQHE. Firstly we show how to derive the topological quantities from the Kubo formula, secondly we recall Laughlin type wave functions with and without periodic boundary conditions and generalize the known results. In the fourth section we show how both concepts fit together, constructing vector bundles from these wave functions. It turns out that the experimentally measured main series (1.1) is described by stable vector bundles. Further, we calculate the fluctuations of the Hall conductance which vanish in the limit of infinite number of particles due of the special form of the Laughlin wave functions. From a physical point of view this explains the independence of the Hall conductivity from the boundary conditions; from a mathematical point of view this is a necessary condition for stability of a vector bundle due to a theorem of S.K. Donaldson [22].

2 From the Kubo formula to Chern Numbers

Following Thouless et. al., we consider a two dimensional interacting electron system in both a magnetic field perpendicular to the plane of area A and an electric field in the x-direction of the plane. Such a system is described by an N-body Hamiltonian

$$H = \sum_{i=1}^{N} \frac{1}{2m} \left(\vec{p}_i - \frac{e}{c} \vec{A} \right)^2 + \sum_{i=1}^{N} U(x_i, y_i) + \sum_{i<j} V(\mid r_i - r_j \mid) \,. \tag{2.1}$$

The Hall current which flows in the y-direction can be obtained via linear response theory by slowly switching on an electric field in x-direction. The first order perturbation expresses the Hall conductivity by the Kubo formula:

$$\sigma_{xy} = \frac{e^2 \hbar}{iA} \sum_{E^\alpha < E_F < E^\beta} \frac{(v^y)_{\alpha\beta}(v^x)_{\beta\alpha} - (v^x)_{\alpha\beta}(v^y)_{\beta\alpha}}{(E^\alpha - E^\beta)^2} \tag{2.2}$$

where E_F is a Fermi energy and the summation implies the sum over all states below and above the Fermi energy. The indices α and β label the bands of the N-body Hamiltonian in the absence of the external electric field. The velocity operator appearing in the Kubo formula is given by

$$\vec{v} = \sum_{i=1}^{N} \frac{1}{m}(\vec{p}_i - \frac{e}{c}\vec{A})\,. \tag{2.3}$$

The Hamiltonian has a symmetry of magnetic translations which are generated by

$$\begin{aligned} k_i^x &= p_i^x - \frac{e}{c}A^x - \frac{e}{c}By_i & (2.4)\\ k_i^y &= p_i^y - \frac{e}{c}A^y + \frac{e}{c}Bx_i \qquad \text{with} & (2.5)\\ [k_i^x, k_i^y] &= \frac{\hbar e}{ic}B\,, & (2.6) \end{aligned}$$

where the magnetic field is given as $B = \partial_x A_y - \partial_y A_x$. Then, the magnetic translations are defined by

$$\begin{aligned} t_i(\vec{L}) &= \exp(\frac{i}{\hbar}\vec{k}_i \cdot \vec{L}) \qquad \text{with} & (2.7)\\ t_i(\vec{a})t_i(\vec{b}) &= t_i(\vec{b})t_i(\vec{a})e^{-i(a\times b)/l^2} & (2.8) \end{aligned}$$

where $l^2 = \frac{\hbar c}{eB}$ is the fundamental length of the system. Thus, the many-body magnetic translation

$$T(\vec{a}) \equiv \prod_{i=1}^{N} t_i(\vec{a}) \tag{2.9}$$

commute with the Hamiltonian for appropriate potential $U(x, y)$.
In order to utilize this symmetry, we impose on the many-body wave function the generalized periodic boundary conditions

$$\begin{aligned} t_i(L_1)\psi &= e^{2\pi i\varphi_1}\psi & (2.10)\\ t_i(L_2)\psi &= e^{2\pi i\varphi_2}\psi\,, & (2.11) \end{aligned}$$

where the parameters φ_1 and φ_2 are independent of the particles indices, as required by the total antisymmetry of the wave function. Now, we perform the gauge transformation

$$\begin{aligned} \psi &\longrightarrow \exp\Big(-\frac{i}{\hbar}\tilde{\varphi}_1(x_1 + \ldots + x_N)\Big) & (2.12)\\ &\qquad \times \exp\Big(-\frac{i}{\hbar}\tilde{\varphi}_2(y_1 + \ldots + y_N)\Big)\psi & (2.13)\\ p_i^x &\longrightarrow p_i^x + \tilde{\varphi}_1 & (2.14)\\ p_i^y &\longrightarrow p_i^y + \tilde{\varphi}_2\,. & (2.15) \end{aligned}$$

It is clear that $\frac{1}{\hbar}\partial\tilde{H}/\partial\tilde{\varphi}_1$ and $\frac{1}{\hbar}\partial\tilde{H}/\partial\tilde{\varphi}_2$ are just the transformed velocity operators of the Kubo formula (2.2) where $\tilde{H}$ is the transformed Hamiltonian H. Thus, after some manipulations the Kubo formula can be written as

$$\sigma_{xy} = \frac{e^2}{i\hbar} \sum_{E_\alpha < E_F < E_\beta} \left(\langle \frac{\partial \psi^\alpha}{\partial \varphi_2} \mid \beta \rangle \langle \beta \mid \frac{\partial \psi^\alpha}{\partial \varphi_2} \rangle - \langle \frac{\partial \psi^\alpha}{\partial \varphi_1} \mid \beta \rangle \langle \beta \mid \frac{\partial \psi^\alpha}{\partial \varphi_1} \rangle \right) . \tag{2.16}$$

Under the assumption that the groundstate ψ_0 is nondegenerate and that regardless of the boundary conditions, there exists a finite energy gap between the groundstate and the exited bulk states – this agrees with the experimentally observed fact that the longitudinal conductance σ_{xx} vanishes at each Hall plateau, since σ_{xx} should be proportional to $\exp\left(\frac{-\Delta}{kT}\right)$ where Δ is the value of the energy gap – the conductance can be further simplified:

$$\sigma_{xy} = \frac{e^2}{\hbar i} \left(\langle \frac{\partial \psi^0}{\partial \varphi_1} \mid \frac{\partial \psi^0}{\partial \varphi_2} \rangle - \langle \frac{\partial \psi^0}{\partial \varphi_2} \mid \frac{\partial \psi^0}{\partial \varphi_1} \rangle \right) . \tag{2.17}$$

So far, we have just performed formal transformations of the Hall conductance without considering its quantization. To understand the latter, we have to equate σ_{xy} with its average over all the phases that specify different boundary conditions.

$$\sigma_{xy} = \frac{e^2}{h} \frac{1}{2\pi i} \int_0^{2\pi} \int_0^{2\pi} d\varphi_1 d\varphi_2 \left(\langle \frac{\partial \psi^0}{\partial \varphi_1} \mid \frac{\partial \psi^0}{\partial \varphi_2} \rangle - \langle \frac{\partial \psi^0}{\partial \varphi_2} \mid \frac{\partial \psi^0}{\partial \varphi_1} \rangle \right) . \tag{2.18}$$

This integral is actually a topological invariant. It is the first Chern number of a $U(1)$ line bundle of the groundstate wave function on the base manifold of a Torus T^2 parameterized by φ_1 and φ_2. Thus, one can draw the important conclusion: on condition that the groundstate is nondegenerate and there exists a finite energy gap, the Hall conductance is always quantized. [10, 13, 14, 15, 12]

If one generalize to a degenerate groundstate of degeneracy r, the equation (2.18) has to be replaced by:

$$\sigma_{xy} = \frac{e^2}{h} \frac{1}{r} \sum_{i=1}^{r} \frac{1}{2\pi i} \int_0^{2\pi} \int_0^{2\pi} d\varphi_1 d\varphi_2 \left(\langle \frac{\partial \psi_i^0}{\partial \varphi_1} \mid \frac{\partial \psi_i^0}{\partial \varphi_2} \rangle - \langle \frac{\partial \psi_i^0}{\partial \varphi_2} \mid \frac{\partial \psi_i^0}{\partial \varphi_1} \rangle \right) , \tag{2.19}$$

where $\{\psi_i\}$ is an orthogonal basis spanning the groundstate Hilbert space [11]. This integral again is a topological invariant, namely, the first Chern number of the determinant bundle of the vector bundle of rank r which is given by this groundstate Hilbert space on the same base manifold as in the nondegenerate case. Thus, the Hall conductance is given as a fraction

$$\sigma_{xy} = \frac{e^2}{h} \frac{c_1}{r} , \tag{2.20}$$

where c_1 is the first Chern number of the above vector bundle. The disadvantage of this approach to the FQHE is that neither the Laughlin wave function for a disk, nor

Haldane's equivalent expression for the wave function on a spherical surface appear to have any degeneracy. Furthermore, these degeneracies are not independent of impurities and disorder. The independence should be respected because impurities an disorder are generally believed to be as essential to the observability of the fractional effect as it is to the integer effect.

3 Laughlin wave functions

In this section we consider the problem of electrons moving in a two dimensional surface in the presence of a perpendicular magnetic field from another point of view following Laughlin [17, 8]. Let us first consider the single electron Hamiltonian

$$H = \frac{1}{2m}(\vec{p} - \frac{e}{c}\vec{A})^2 = \frac{1}{2}m(v_x^2 + v_y^2) \tag{3.1}$$

with constant magnetic field $B = \partial_x A_y - \partial_y A_x$. Since the velocity operator

$$\vec{v} = \frac{1}{m}(\vec{p} - \frac{e}{c}\vec{A}) \tag{3.2}$$

fulfills the following commutation relation

$$[v_x, v_y] = \frac{\hbar\omega_c}{m} i \qquad \text{with} \qquad \omega_c = \frac{eB}{mc} \tag{3.3}$$

the energy spectrum of this Hamiltonian is the same as that of the harmonic oscillator:

$$E_n = \hbar\omega_c(n + \frac{1}{2})\,. \tag{3.4}$$

Each level, called Landau level, is infinitely degenerate due to the rotational symmetry around the z-axes. In a finite sample of area A, one can show that the degeneracy of each Landau level is determined by the number of magnetic flux quanta

$$N_S = \frac{BA}{\Phi_0} = \frac{\Phi_{mag}}{\Phi_0} \tag{3.5}$$

where Φ_{mag} is the magnetic flux through the area A and $\Phi_0 = \frac{hc}{e}$ is a single flux quantum. In the Landau gauge the wave functions in the lowest Landau level (LLL) $(n = 0)$ are generated by

$$\psi(z) \sim z^k \exp(\frac{1}{4l^2}\,|\,z\,|^2)\,, \qquad k = 0 \ldots N_S - 1\,, \qquad z = x + iy \qquad l^2 = \frac{\hbar c}{eB}\,. \tag{3.6}$$

Let us now consider the case of N such electrons. If there is no interaction between them, the many-particle problem splits into N copies of the single particle problem. Since the magnetic field B controls the number of states and thus the density of electrons per

state, its action can be considered as an external pressure. The electron density per state i .e. the filling factor is defined as ν:

$$\nu = \frac{N}{N_S}\,. \tag{3.7}$$

Laughlin was the first to realize that the many-particle groundstate of the QHE is fundamentally different from other kown condensed states, describing magnetism or superconductivity. For the FQHE, with filling fraction $\nu = \frac{1}{2p+1}$ he found by numerical experiments, the groundstates to be given by the following wave functions:

$$\psi(z_1, \ldots, z_N) \sim \prod_{i<j} (z_i - z_j)^{2p+1} \exp(-\frac{1}{4l^2} \sum_i \mid z_i \mid^2) \tag{3.8}$$

where p should be an integer in order to make ψ obey the Pauli principle. The filling factor ν for these wave functions can be determined with the help of the following argument from (3.6): the highest power of z_i plus one is exactly the number of possible states which can be occupied by one electron. In the case of the Laughlin wave functions (3.8) this is $N_S = (2p+1)(N-1)+1$ for each electron z_i. In the limit of infinitely many particles the filling fraction ν converges to $1/(2p+1)$. Extensive calculations have proven these wave functions to be extremely close to the numerical exact solution.
Generalizations of these wave functions to other filling fractions exist [26, 27, 25, 24]. These are Laughlin type wave functions of the following form:

$$\psi_K(\{z_i^I\}) \sim \prod_{\substack{I\\ i<j}} (z_i^I - z_j^I)^{K_{II}} \prod_{\substack{I<J\\ i\leq j}} (z_i^I - z_j^J)^{K_{IJ}} \exp(-\frac{1}{4l^2} \sum_{I,i} \mid z_i^I \mid^2) \tag{3.9}$$

where now the electrons are distributed to n different subbands which are labeled by $I, J = 1 \ldots n$ each subband I containing N_I electrons. To assume the validity of the Pauli principle and the single valuedness of the wave function K_{IJ} should be a symmetric, positive, integer valued matrix with odd integers on the main diagonal. These different subbands are interpreted as different layers or different Landau levels or as additional quantum numbers in the first Landau level, which can also depend on the third coordinate. The highest power of z_i^I plus one gives the number of states which can be occupied by the electron:

$$K_{II}(N_I - 1) + \sum_{I\neq J} K_{IJ} N_J = \sum_J K_{IJ} N_J - K_{II} \tag{3.10}$$

The magnetic field determines the number of possible states, which is exactly the number of flux quanta N_Φ, so:

$$N_\Phi = \sum_J K_{IJ} N_J - K_{II} \longrightarrow \sum_J K_{IJ} N_J \qquad \text{for } N_I, N_\Phi \text{ large} \tag{3.11}$$

It follows: $N_I = \sum_J (K^{-1})_{IJ} N_\Phi$. Thus, the filling fraction is now given by:

$$\nu = \frac{\sum_I N_I}{N_\Phi} = \sum_{IJ} (K^{-1})_{IJ} \tag{3.12}$$

F.D. Haldane and E.H. Rezayi have constructed Laughlin's wave functions in the Landau gauge with periodic boundary conditions for the filling fraction $\nu = \frac{1}{m}$. Later E. Keski-Vakkuri and X.G. Wen have generalized this to the Laughlin type wave functions of the form (3.9).
Following Haldane and Rezayi, the wave function describing a particle confined to the lowest Landau level has the analytic form in the Landau gauge ($\vec{A} = -Bye_x$)

$$\psi(x,y) = \exp(-\frac{1}{2l^2}y^2)f(z), \qquad z = x + iy \tag{3.13}$$

where $f(z)$ is a holomorphic function. The periodic boundary condition on the wave function (2.11) is given by (with $\tau = L_2 e^{i\theta}$)

$$\begin{aligned} f(z+L_1) &= e^{2\pi i\varphi_1} f(z) &(3.14)\\ f(z+L_2e^{i\theta}) &= e^{2\pi i\varphi_2}\exp(i\pi N_s[(2z/L_1)+\tau])f(z) &(3.15)\end{aligned}$$

Since $f(z)$ is a holomorphic function, the integral of $\frac{f'(z)}{f(z)}$ along the boundaries requires the number of zeros to be precisely N_s. The possible analytic form of $f(z)$ is thus strongly constrained, and the most general form is expressible as:

$$\begin{aligned} f(z) &= \exp(ikz)\prod_{n=1}^{N_s}\vartheta_1\left((z-z_n)/L_1 \mid \tau\right) &(3.16)\\ \vartheta_1(z\mid\tau) &= \sum_n \exp(i\tau\pi(n+\frac{1}{2}) + 2\pi i(n+\frac{1}{2})(z+\frac{1}{2}))\,. &(3.17)\end{aligned}$$

There are some additional constraints on k and the z_n that guarantee the number of linearly independent solutions to be equal to the number of zeros of $f(z)$ [20].
Let us now consider the many-particle wave functions. F.D.M. Haldane and E.H. Rezayi have first used translational invariance to express those as a product of a center-of-mass term and a factor involving only relative coordinates. Following R.B. Laughlin they have written down the groundstate wave function for $\nu = 1/m$ with periodic boundary conditions (2.11); this has been generalized to other rational filling factors by E. Keski-Vakkuri and X.G. Wen for periodic boundary conditions with $\varphi_1 = 0$ and $\varphi_2 = 0$: They expressed the groundstate wave function in the following way:

$$\psi_K(\{z_i^I\}) = F^{c.m.}(Z_1,\ldots,Z_N)\prod_{\substack{I\\ i<j}}\vartheta_1(z_i^I - z_j^I)^{K_{II}}\prod_{\substack{I<J\\ i\leq j}}\vartheta_1(z_i^I - z_j^J)^{K_{IJ}} \tag{3.18}$$

$$\exp\left(-\frac{1}{2l^2}\sum_{I,i}(y_i^I)^2\right) \tag{3.19}$$

where the Z_I are the center of mass coordinates:

$$Z_I := \sum_{k=1}^{N_I} z_k^I\,. \tag{3.20}$$

From the quasiperiodicity of the ϑ_1-functions they derived the following conditions on the center of mass functions

$$
\begin{aligned}
F^{c.m.}(Z^I+1) &= F^{c.m}(Z^I) && (3.21)\\
F^{c.m.}(Z^I+\tau) &= \exp(-i\pi\tau K_{II} - 2\pi i\sum_J K_{IJ}Z_J)F^{c.m}(Z^I) && (3.22)
\end{aligned}
$$

and got $|\det(K)|$ independent solutions for $F^{c.m.}(Z_1,\ldots,Z_n)$ which gives a degeneracy of the groundstate of order $|det(K)|$.

In order to study the influence on the boundary conditions to the degenerate groundstate, we generalize the quasi periodicity conditions and get for the center of mass functions:

$$
\begin{aligned}
F^{c.m.}(Z^I+1) &= \exp(2\pi i\varphi_1)F^{c.m}(Z^I) && (3.23)\\
F^{c.m.}(Z^I+\tau) &= \exp(2\pi i\varphi_2)\exp(-i\pi\tau K_{II} - 2\pi i\sum_J K_{IJ}Z_J)F^{c.m}(Z^I)\,. && (3.24)
\end{aligned}
$$

The $|\det(K)|$ solutions of $F^{c.m.}$ can be calculated in the same way as before leading to the following expression:

$$
\begin{aligned}
F^{c.m.}_{\vec{\alpha}}(\vec{Z}) = \sum_{\vec{m}\in Z^n} &\quad \exp\left(i\pi\tau\vec{m}^tK\vec{m} + 2\pi i\tau\vec{m}(\vec{\alpha}+\varphi_1\vec{e}) - 2\pi i\vec{m}\varphi_2\vec{e}\right) && (3.25)\\
\times &\quad \exp\left(2\pi i(\vec{\alpha}+K\vec{m}+\varphi_1\vec{e})\,\vec{Z}\right), && (3.26)
\end{aligned}
$$

where $\vec{e}$ is the vector with entries all equal 1 and $\vec{\alpha}$ is a vector that characterizes the groundstate and belongs to the coset $\mathbb{Z}^n/K\mathbb{Z}^n$ which has exactly $|\det(K)|$ elements.

Let us make two remarks: Firstly, E. Keski-Vakkuri and X.G. Wen also showed [21] that the center-of-mass part $F^{c.m.}_{\vec{\alpha}}(\vec{Z})$ can be considered as the degenerate groundstate of a mean field theory for the QHE, which is given by the following Chern-Simons Lagrangian:

$$
\mathcal{L} = \frac{1}{4\pi}\sum_{I,J} K_{IJ}\epsilon^{\mu\nu\lambda}a_{I\mu}\partial_\nu a_{J\lambda} \tag{3.27}
$$

This way of describing the QHE has been extensively studied [29, 30, 31].

Moreover, the form of the $F^{c.m.}_{\vec{\alpha}}(\vec{Z})$ does not depend on the explicit structure of the wave function, but only on the number and degree of zeros of the wave function. This seems to be similar to the Hofstadter problem which was recently reformulated by P.B. Wiegmann and A.V. Zabrodin [28] who showed that the zeros of these wave function fulfill some Bethe-Ansatz equations.

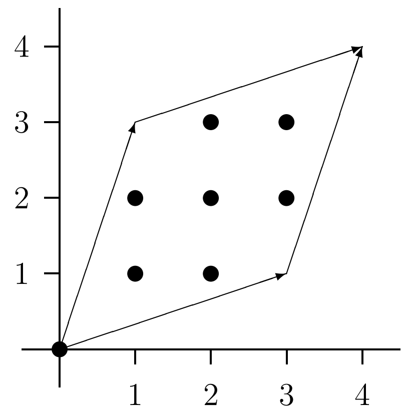
Finally, let us give an example.

$$
K = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \Rightarrow K^{-1} = \frac{1}{8}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \tag{3.28}
$$

Thus, $\det(K) = 8$, $\nu = \frac{1}{8}(3-1+3-1) = \frac{1}{2}$ and the vectors $\vec{\alpha}$ which label the eight-fold degenerate groundstate can be written in the following way:

$$
\vec{\alpha} \in \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \end{pmatrix} \right\} \tag{3.29}
$$

This may be represented graphically as follows.



From now on let us assume that each subband contains the same number of particles: $N_I = N_J = N, \quad I, J = 1, \ldots, n$. From equation (3.11) it follows, that $\vec{e}$ must be an eigenvector corresponding to K. Further, let us make the naturally restriction to matrices K which are invariant under arbitrary permutations of all subbands. This symmetry implies that $K_{IJ} = K_{\sigma(I)\sigma(J)}$ where σ denotes an arbitrary permutation of $\{1, \ldots, n\}$.

4 Vector bundles over the Torus

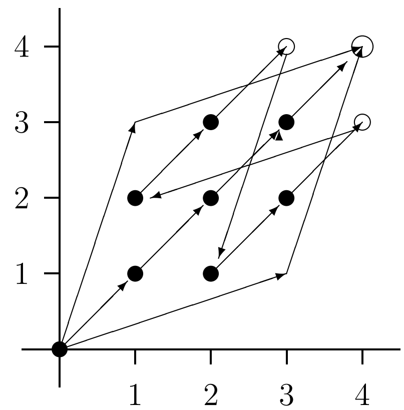
Next, we want to construct a nontrivial vector bundle, which has the torus parameterized by φ_1 and φ_2 as base manifold, from the generalized Laughlin wave functions with periodic boundary conditions constructed in the last section. Further, the filling factor can now be expressed as a topological quantity, the first Chern number of the vector bundle divided by its rank.
In the wave functions of the last section for a given matrix K only the center-of-mass part of the wave function depends on φ_1 and φ_2. Thus this part alone defines a vector bundle by transition functions in the following way:

$$\varphi_1 \to \varphi_1 + 1 \quad : \quad F^{c.m.}_{\vec{\alpha}}(\vec{Z}) \to F^{c.m.}_{\vec{\alpha}+\vec{e}}(\vec{Z}) \tag{4.1}$$

$$\varphi_2 \to \varphi_2 + 1 \quad : \quad F^{c.m.}_{\vec{\alpha}}(\vec{Z}) \to F^{c.m.}_{\vec{\alpha}}(\vec{Z}) \tag{4.2}$$

$$F^{c.m.}_{\vec{\alpha}+\vec{K}_I}(\vec{Z}) = \exp(\pi K_{II} + 2\pi\alpha_I + 2\pi(\varphi_1 + i\varphi_2))F^{c.m.}_{\vec{\alpha}}(\vec{Z})\,, \tag{4.3}$$

where $\vec{K}_I$ is a column of the K. The rank r of this vector bundle is given by $|\det(K)|$. Refering to the example of the last section, the action of φ_1 can be represented graphically as follows:



The Chern number or degree of the vector bundle can be determined by counting the zeros of the corresponding determinant bundle which is given by a product over a basis $\{F^{c.m.}_{\vec{\alpha}_1}, \ldots, F^{c.m.}_{\vec{\alpha}_{\det(K)}}\}$. Then it is easy to see that the first Chern number is given by

$$c_1 = |\det(K)|\, |\sum_{I,J} (K^{-1})_{I,J}| \ . \tag{4.4}$$

Thus, the topological quantity

$$\mu = \frac{c_1}{r} \tag{4.5}$$

coincides with filling factor since ν is always positive if $\vec{e}$ is an eigenvector of K (see equation (3.12)).

In the following, we define an indecomposable vector bundle E as a vector bundle which can not be written as the direct sum of two other bundles E_1 and E_2 over the same base manifold M: $E = E_1 \oplus E_2$. A vector bundle is called simple if $\dim_{\mathbb{C}} H^0(M, \mathrm{End}(E)) = 1$. If E is a simple vector bundle of rank n over M then E is indecomposable [32]. Studying now these vector bundles in more detail, it is not so difficult to calculate that $\dim_{\mathbb{C}} H^0(M, \mathrm{End}(E)) = \gcd(c_1, r)$. Thus, the indecomposable vector bundles are determined by the condition $\gcd(c_1, r) = 1$ and given by the following matrices

$$K = \begin{pmatrix} 2p \pm 1 & 2p & \ldots & 2p \\ 2p & \ddots & & \vdots \\ \vdots & & & 2p \\ 2p & \ldots & 2p & 2p \pm 1 \end{pmatrix} \tag{4.6}$$

with filling fraction

$$\nu = \frac{n}{2pn \pm 1} \, . \tag{4.7}$$

Most remarkably, these are just half of the fractional values which are found experimentally (1.1). The electron-hole duality which transforms $\nu \to 1-\nu$ gives the other half of (1.1). Thus, we have found a topological argument which selects the right fractional filling factors ν for the FQHE.
In the second section we have seen that the Hall conductivity is calculated by averaging over the different boundary conditions. There is also another equivalent definition of the Chern number not counting the zeros of the determinant bundle but using the curvature of the vector bundle:

$$c_1 = \frac{1}{2\pi i}\int d\varphi^2 \mathrm{tr}(F)\,. \tag{4.8}$$

The curvature is determined by the hermitian structure which is physical naturally given by the scalar product which we already used in the second section. Comparing with equation (2.19), F is given by

$$F^{\vec{\alpha}\vec{\beta}} = \langle \frac{\partial \psi_{\vec{\alpha}}}{\partial \varphi_1} \mid \frac{\partial \psi_{\vec{\beta}}}{\partial \varphi_2} \rangle - \langle \frac{\partial \psi_{\vec{\alpha}}}{\partial \varphi_2} \mid \frac{\partial \psi_{\vec{\beta}}}{\partial \varphi_1} \rangle \tag{4.9}$$

where now $\psi_{\vec{\alpha}}$ are the normalized wave functions. The explicit calculation of $\mathrm{tr}(F^{\vec{\alpha}\vec{\beta}})$ is very long and tedious. One has to integrate over all electron coordinates $\{z_i^I\}$. But only if one takes the whole electron wave function and not only the center of mass term which already determines the vector bundle it is possible to find an explicit expression for $F^{\vec{\alpha}\vec{\beta}}$. This is technically very difficult since one has to handle with a lot of sums coming from the ϑ_1-functions in the wave function as can be seen in the Appendix A, but most surprisingly, it is possible to obtain an explicit expression. For $n=1$ and $K=(p)$ and more than two particles (i.e. $N>1$) $\mathrm{tr}(F^{\vec{\alpha}\vec{\beta}})$ can be expressed in the following way:

$$\begin{aligned} \mathrm{tr}(F^{\vec{\alpha}\vec{\beta}}) &= \frac{1}{Z}\sum_{r_1,\ldots,r_{N-1}=0}^{2pN-1} \alpha(r_1,\ldots,r_{N-1}) \times \\ & \sqrt{\frac{2}{pN}}\sum_k \sum_{j=1}^{N} \frac{1}{N}\exp\left(-\frac{2\pi}{pN}(\varphi_1+k+r_j)^2\right) \end{aligned} \tag{4.10}$$

(4.11)

where the $\alpha(r_1,\ldots,r_{N-1})$ and Z are given in the appendix A. If K is of the form (4.6) $\mathrm{tr}(F^{\vec{\alpha}\vec{\beta}})$ can be expressed in a quite similar way, p has to be replaced by λ, the eigenvalue of K by the eigenvector $\vec{e}$. $\mathrm{tr}(F^{\vec{\alpha}\vec{\beta}})$ is independent of φ_2 for more than two particles, but if it is plotted as a function of φ_1 one sees that it oscillates or fluctuates around a mean value. However, it is very remarkable that these fluctuations vanish exponentially with the number of particles. That means in the limit of an infinite particle number $\mathrm{tr}(F^{\vec{\alpha}\vec{\beta}})$ is independent of φ_1 and φ_2 and equals $2\pi i c_1$. This is an important observation, since it explains the independence of the QHE of boundary conditions and therefore the equality of exact measurements for different sample geometries. This also justifies the averaging of

the Hall conductance in the second section in order to express it as the first chern number of a vector bundle.
From a mathematical point of view, the quantity ν is the most important quantity studying stable vector bundles. Stability is an useful topological property to restrict the moduli space of certain vector bundles. A vector bundle E is called stable if for every proper subbundle F the relation $\mu(F) < \mu(E)$ is true (or semi-stable if $\mu(F) \leq \mu(E)$). Generally it is very difficult to check this property. But there is a theorem of Donaldson that an indecomposable vector bundle over an Riemannian surface is stable if and only if there exists a metric on the vector bundle such that the trace of the curvature is constant [22]. This metric is unique up to scale factors. In our cases on the torus stability and indecomposability are equivalent.
It is very astonishing that just the Laughlin wave functions which have been found by numerical studies give the right hermitian structure for our vector bundles, since only for them in the limit $N \to \infty$ $\mathrm{tr}(F^{\vec{\alpha}\vec{\beta}})$ converges to a constant function.
This shows a very nice interplay between experimental physics, theoretical physics and pure mathematics.

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A Appendix

In the following, the multi summation indices are understand to have the properties:

$$
\begin{aligned}
n_r^{ij} &= -n_r^{ji}\,, \qquad 1 \le i,j \le N\,, \qquad r = 1\ldots p && \text{(A.1)}\\
l_r^{ij} &= -l_r^{ji}\,, \qquad 1 \le i,j \le N\,, \qquad r = 1\ldots p && \text{(A.2)}\\
n^{ij} &= \sum_{r=1}^{p} n_r^{ij} && \text{(A.3)}\\
l^{ij} &= \sum_{r=1}^{p} l_r^{ij} && \text{(A.4)}\\
g^{i} &= \sum_{j\neq i} (n^{ij} + l^{ij}) && \text{(A.5)}\\
\mathrm{erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt\,. && \text{(A.6)}
\end{aligned}
$$

Then

$$
\begin{aligned}
<\tilde{\psi}_\alpha\,|\,\tilde{\psi}_\beta> \;&=\; \delta_{\alpha\beta} \left(\frac{1}{8pN}\right)^{\frac{N}{2}} \exp\left(\frac{2\pi}{p}(\alpha+\varphi_1)^2\right) \times \\
\sum_{\substack{m\,\in\,\mathbf{Z} \\ n_1^{ij},\ldots,n_p^{ij}\,\in\,\mathbf{Z}+\frac{1}{2} \\ l_1^{ij},\ldots,l_p^{ij}\,\in\,\mathbf{Z}+\frac{1}{2} \\ 1\le i<j\le N \\ \sum_{j=1}^{N}(n^{ij}-l^{ij})=0}} \quad & \exp\Big(-\pi \sum_{\substack{i<j\\ r}} \Big((n_r^{ij})^2 + (l_r^{ij})^2\Big) - \tfrac{\pi}{2Np}\Big(\sum_i (g_i)^2\Big)\Big) \\
& \times \prod_{i=1}^{N} \Big(\mathrm{erf}\Big(\tfrac{\pi}{2pN}\big(2(\alpha+\varphi_1) + 2p(m+N) + g_i\big)\Big) \\
& \qquad -\mathrm{erf}\Big(\tfrac{\pi}{2pN}\big(2(\alpha+\varphi_1) + 2pm + g_i\big)\Big)\Big)
\end{aligned}
$$

$$
\Rightarrow \qquad <\tilde{\psi}_\alpha\,|\,\tilde{\psi}_\beta> = \delta_{\alpha\beta} \frac{N}{(2pN)^{\frac{N}{2}}} \exp\left(\frac{2\pi}{p}(\alpha+\varphi_1)^2\right) \sum_{r_1,\ldots,r_{N-1}=0}^{2pN-1} \alpha(r_1,\ldots,r_{N-1}) \qquad \text{(A.7)}
$$

where

$$\alpha(r_1,\ldots,r_{N-1}) = \sum_{\substack{n_1^{ij},\ldots,n_p^{ij}\\ l_1^{ij},\ldots,l_p^{ij}\\ 1\le i<j\le N\\ n_1^{1i}=0\\ \sum_{j=1}^N (n^{ij}-l^{ij})=0\\ g_i = r_i (\mathrm{mod}\ 2pN)}} \exp\Big(-\pi\sum_{\substack{i<j\\ r}}((n_r^{ij})^2+(l_r^{ij})^2) - \frac{\pi}{2Np}(\sum_i g_i^2)\Big) \quad \text{(A.8)}$$

and

$$Z := \sum_{r_1,\ldots,r_{N-1}=0}^{2pN-1} \alpha(r_1,\ldots,r_{N-1}) \quad \text{(A.9)}$$

With the definition of the connection

$$A_1^{\alpha\beta} = <\psi_\alpha|\partial_{\varphi_1}\psi_\beta> \quad \text{(A.10)}$$

$$A_2^{\alpha\beta} = <\psi_\alpha|\partial_{\varphi_2}\psi_\beta> \quad \text{(A.11)}$$

$$F^{\alpha\beta} = \partial_{\varphi_1}A_2^{\alpha\beta} - \partial_{\varphi_2}A_1^{\alpha\beta} \quad \text{(A.12)}$$

one gets

$$\begin{aligned} \frac{1}{2\pi i}\mathrm{tr}(\partial_{\varphi_1}A_2^{\alpha\beta}) &= \frac{1}{Z}\sum_{r_1,\ldots,r_{N-1}=0}^{2pN-1}\alpha(r_1,\ldots,r_{N-1})\times \\ &\quad \sqrt{\frac{2}{pN}}\sum_k\sum_{j=1}^N \frac{1}{N}\exp\left(-\frac{2\pi}{pN}(\varphi_1+k+\frac{r_j}{2})^2\right) \quad \text{(A.13)} \\ \text{with} \quad r_N &= -r_1-\ldots-r_{N-1} \\ \frac{1}{2\pi i}\mathrm{tr}(\partial_{\varphi_2}A_1^{\alpha\beta}) &= 0 \qquad \text{if} \qquad N\ge 2 \quad \text{(A.14)} \end{aligned}$$

and

$$\begin{aligned} c_1 &= \frac{1}{2\pi i}\int d^2\varphi\ \mathrm{tr}(F^{\alpha\beta}) = \frac{1}{2N}\frac{1}{Z}\sum_{r_1,\ldots,r_{N-1}=0}^{2pN-1}\alpha(r_1,\ldots,r_{N-1})\times \\ &\quad \sum_k\sum_{j=1}^N \Big(\mathrm{erf}(\sqrt{\frac{2\pi}{pN}}(k+1+\frac{r_j}{2})) - \mathrm{erf}(\sqrt{\frac{2\pi}{pN}}(k+1+\frac{r_j}{2}))\Big) \\ &= \frac{1}{Z}\sum_{r_1,\ldots,r_{N-1}=0}^{2pN-1}\alpha(r_1,\ldots,r_{N-1}) = 1 \quad \text{(A.15)} \end{aligned}$$

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